Identification of Causal Models with Unobservables: A Self-Report Approach

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Abstract

This paper presents a novel self-report approach to identify a general causal model with an unobserved covariate, which can be unobserved heterogeneity or an unobserved choice variable. It shows that a carefully designed noninvasive survey procedure can provide enough information to identify the complete causal model through the joint distribution of the observables and the unobservable. The global nonparametric point identification results provide sufficient conditions under which the joint distribution of four observables, two in a causal model and two from surveys, uniquely determines the joint distribution of the unobservable in the causal model and the four observables. The identification of such a joint distribution including the unobserved covariate implies that the complete causal model is identified.

JEL classification: *C01, C14*
Keywords: *Causal model, Measurement error model, Nonparametric identification*

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1 Introduction

In a complete causal model, we are interested in the impact of an agent's behavior or characteristics, i.e., explanatory variables, on an outcome Y. The explanatory variables include a variable X which is observed in a sample and an unobservable U. The complete causal model can be described by a conditional distribution [1]

$$f_{Y|X,U}$$

or equivalently a function $Y = h(X, U, \epsilon)$, where ϵ is a white noise. Inherently, the outcome should be realized after the explanatory variables are realized, i.e., $t_2 > t_1$ in Figure 1. In this paper, I propose a novel self-report approach to identify the complete model by identifying the joint distribution $f_{Y,X,U}$.

An explanatory variable, X or U, may be an individual characteristic or a choice variable. The former includes, for example, age, race, ability or risk aversion level, and the latter may be program participation decision, education level, or effort level. Here we consider four possible cases without specifying the causality between X and U:

- Case 1: X is a choice variable and U is an unobserved heterogeneity or characteristics. This is a typical treatment effect model with unobserved heterogeneity. For example, X can be education or program participation and U can be ability or risk aversion level.

- Case 2: Both X and U are a choice variable. For example, U can be an effort level or a subjective belief.

- Case 3: U is a choice variable and X is an observed characteristics. For example, X can be race, gender, or family background.

- Case 4: Both X and U are an individual characteristics.

A typical example in Case 1 includes X as an indicator of a treatment choice with $X = x_1$ standing for being treated, x_0 otherwise. In the complete model describing the causal relationship between Y and (X, U), the causal effect is defined as

$$\mathbf{CE}(U) = E(Y|X = x_1, U) - E(Y|X = x_0, U).$$

This causal effect $\mathbf{CE}(U)$ can be directly estimated when we know the joint distribution of $f(Y, X, U)$.

When U is unobserved, the causal model becomes incomplete and X becomes endogenous. The average treatment effect (ATE) defined as $E_U[\mathbf{CE}(U)]$, where expectation E_U is with

[1] We use $f_{A|B}$ to stand for the conditional distribution function of variable A conditional on variable B.

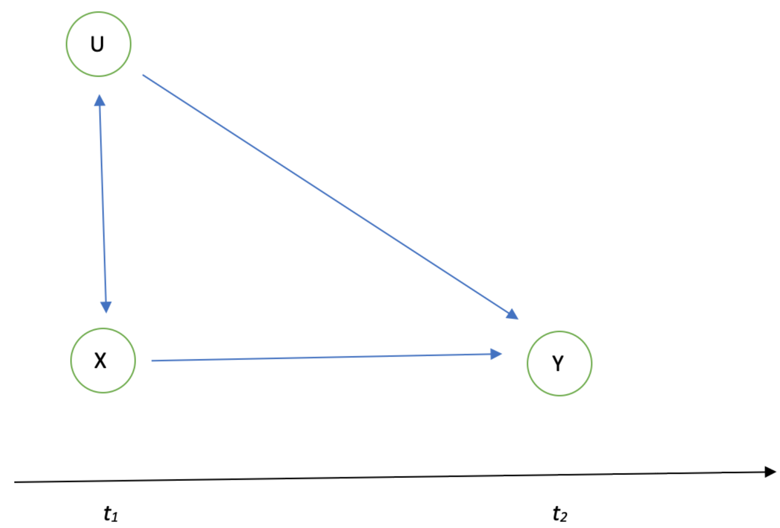

Figure 1: A causal model with an unobservable

respect to the marginal distribution of U. Furthermore, the average treatment effect on the treated (ATET) becomes $E_{U|X=x_1}[\mathbf{CE}(U)]$ where expectation $E_{U|X=x_1}$ is with respect to distribution of U conditional on being treated, i.e., $X = x_1$.

The existing literature on treatment effects is based on the randomization approach, which is widely adopted in the biostatistic and medical research. It eliminates the correlation between X and U through a direct randomization, an indirect randomization (instrument variables), a conditional randomization (unconfounded assignments), a local randomization (regression discontinuity), or a second-order randomization (difference-in-difference). This approach focuses on the ATE and the ATET without estimating the complete model. In addition, the randomization approach only applies to cases 1 and 2, where the endogenous X is a choice variable. I refer to Imbens and Rubin (2015), Pearl (2009) and Heckman and Vytlacil (2007a,b) for a review of the huge literature on treatment effects in economics, biostatistics, and other disciplines.

In this paper, I propose a self-report approach to identification of the complete model by identifying the joint distribution $f_{Y,X,U}$. Apparently, such an identification leads to identification of all the treatment effects above, i.e., ATE and ATET. Instead of taking the popular approach in biomedical research, we noninvasively measure the unobservable in the model through surveys and then identify a complete model as in physics and chemistry. Given that we have developed powerful tools to handle self-reporting errors in survey data, [2] I propose a self-report procedure through surveys to collect more information on unobservables to iden-

[2]For reviews of this extensive literature, we refer to Wansbeek and Meijer (2000), Bound et al. (2001), Fuller (2009), Chen et al. (2011), Carroll et al. (2012), Schennach (2016), Hu (2017), Schennach (2019), and Hu (2021).

tify the complete model. This self-report survey procedure will be guided by a model in mind in the sense that if a model can guide researchers on what U is about, then it is more likely to design surveys satisfying the conditions we need. In that case, the proposed approach makes use of some information from a structural model without further specification. Nevertheless, A key assumption in such a self-report approach is that the self-report procedure in surveys will not intervene the causal relationship in the complete model.

We design the survey questions under the belief that an individual characteristics, observed or unobserved, will generally affect all the answers to survey questions, and that a choice variable will only affect answers to survey questions about the choice. Therefore, if U is an unobserved heterogeneity, we should expect that all the measurements, i.e., survey answers, are a function of U. If U is an unobserved choice variable, we will need a model to guide us on what U is about and design a question targeting at it. In Case 1, which is widely used in the causal inference literature, I propose to measure the unobserved heterogeneity U before and after the outcome is realized. Because X is a choice variable, one can design the second measurement such that it does not depend on the choice X but depends on outcome Y and U. When both X and U are a choice variable as in Case 2, we will need a model to guide us on how to design a survey question about U. But the self-report procedure and the identification strategy for Case 1 still applied. In Case 3, where U is a choice variable and X is an observed characteristics, X will affect all the survey answers. We not only need a model to design a measurement targeting at U, but also a different identification strategy, i.e., repeated measurements before the outcome is realized. When both U and X are individual characteristics as in Case 4, the repeated measurement procedure still applies. In summary, if the measurements are well designed and don't change the causal relationship of interest, we are able to identify the complete model in all the four cases above.

This paper is organized as follows: Second 2 introduces the self-report approach; Section 3 provides the key identification results; Section 4 shows an alternative self-report procedure; A summary is in Section 5 and proofs are in Appendix.

2 A Self-Report Approach

In the benchmark setting, we consider the case where X is a choice variable as in Cases 1 and 2. Inherently, the explanatory variables X and U should be realized before the outcome Y does as shown in Figure 1. For example, an economic agent makes a choice X at time t_1 and the outcome Y is realized at time t_2 in case 1 with $t_2 > t_1$. I propose a self-report survey procedure as follows:

1. Between t_1 and t_2, we take a measurement Z of U by asking a question related to X.

2. After t_2, we take a measurement W of U by asking a question related to Y,

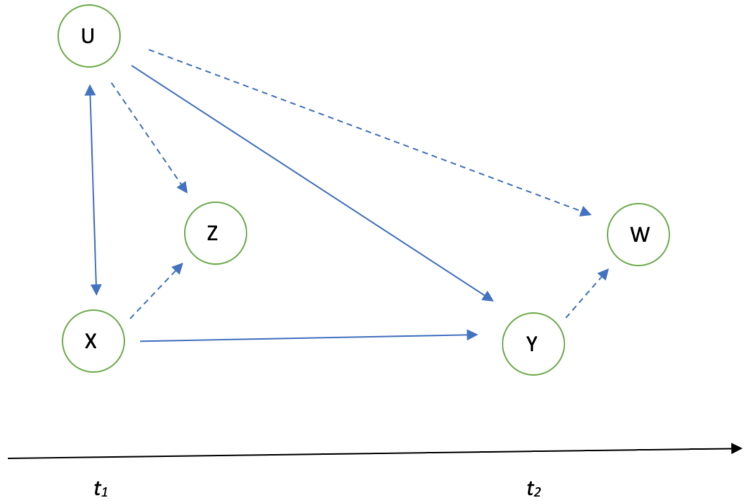

Figure 2: Measurements of an unobservable in a causal model. Measurements are “caused by” the variables in the model. The measurement procedure should not intervene with the causality among the variables in the model, i.e., outcome Y and explanatory variables X and U.

How to specify survey questions will be guided by a model, which should show what the unobservable U is about. In Case I, where X is a choice variable and U is an unobserved heterogeneity, the first survey may ask “Why did you choose X?” and the second may have “What impact do you expect from Y?” In Case 2 and Case 3, where U is, say, a choice of an effort level, the first survey may ask “How much effort did you make given X?”. In Case 4, where U is an ability level, the first survey may ask about previous test scores.

The surveys should not intervene the causal relationship in the complete model $f_{Y|X,U}$ itself. And given the timing structure of the explanatory variables and the outcome, as shown in Figure 2, the measurement procedure intends to guarantee that measurement W only depends on outcome Y, unobserved U and an conditionally independent measurement error and that measurement Z is a function of choice X, unobserved U, and another measurement error. The measurement errors need to satisfy assumptions as follows:

Assumption 1 *(Conditional independence) The two measurements, Z and W, satisfy:*

$$f_{W|Y,X,Z,U} = f_{W|Y,U} \tag{1}$$

$$f_{Y|Z,X,U} = f_{Y|X,U} \tag{2}$$

Assumption 1 implies that how an agent answers the first survey will not affect the causality in the model and that the agent will only consider what is being asked in the second survey, i.e., U and Y, instead of X. This assumption is particularly suitable for the widely studied scenario as in Case 1, where X is a choice variable and U is the unobserved heterogeneity. Section 4 will provide another self-report procedure more suitable for the case where X is the individual characteristics. The self-reporting errors don't need to be classical, i.e, additive to and independent of the true values, but need to satisfy the conditional independence so that

$$\begin{aligned} f_{W,Y,Z,X,U} &= f_{W|Y,Z,X,U} f_{Y|Z,X,U} f_{Z,X,U} \\ &= f_{W|Y,U} f_{Y|X,U} f_{Z,X,U}. \end{aligned}$$

We then present the sufficient conditions under which the joint distribution of observables and unobservables, i.e., $f_{W,Y,Z,X,U}$, is uniquely determined by the distribution of observables $f_{W,Y,Z,X}$.

Given that the randomization approach and the self-report approach can both identify and estimate the ATE and the ATET, the comparison between estimates from two approaches can provide a test on the key conditional independence in Assumption 1. In that sense, the randomization approach is still the gold standard. Furthermore, researchers can adjust the self-report procedure such that the ATE and ATET estimates from the self-report approach are consistent with those from the randomization approach. With a validated self-report procedure, the new approach will be able to reveal complete causal models.

3 Nonparametric Identification

For simplicity of the analysis, we focus on the discrete case. The results can be extended to the case with a continuous U with the same intuition. We assume

Assumption 2 *The two measurements, Z and W, and the unobservable U share the same known support $\mathcal{U} = \{1, 2, ..., K\}$.*

Here we assume K is known. In fact, if the support of measurements Z and W are large enough, we can identify K from the rank of an observed matrix under conditional independence. Since this is not a main focus of this paper, we simply assume K is known. The observed distribution is associated with the unknown ones as follows:

$$f_{W,Y,Z,X} = \sum_{U} f_{W|Y,U} f_{Y|X,U} f_{Z,X,U} \tag{3}$$

Inspired by the identification strategy in Carroll et al. (2010) and Hu and Shum (2012), we define for any fixed (y, x)

$$
\begin{aligned}
M_{W,y,Z,x} &= [f_{W,Y,Z,X}(i,y,j,x)]_{i=1,2,...,K;j=1,2,...,K} \quad (4)\\
M_{W|y,U} &= \left[f_{W|Y,U}(i|y,j)\right]_{i=1,2,...,K;j=1,2,...,K}.\\
M_{Z,x,U} &= [f_{Z,X,U}(i,x,j)]_{i=1,2,...,K;j=1,2,...,K}.\\
D_{y|x,U} &= \begin{bmatrix} f_{Y|X,U}(y|x,1) & 0 & 0 \\ 0 & ... & 0 \\ 0 & 0 & f_{Y|X,U}(y|x,K) \end{bmatrix} \quad (5)
\end{aligned}
$$

Equation (3) then implies

$$M_{W,y,Z,x} = M_{W|y,U} D_{y|x,U} (M_{Z,x,U})^T \quad (6)$$

where superscript T stands for matrix transpose. Our identification results rely on a key invertibility assumption as follows:

Assumption 3 *(Matrix invertibility) for any $y \in \mathcal{Y}$, there exists a $(x, \overline{x}, \overline{y})$ such that i) $M_{W,y,Z,x}$, $M_{W,\overline{y},Z,x}$, $M_{W,\overline{y},Z,\overline{x}}$, and $M_{W,y,Z,\overline{x}}$ are invertible and ii) for all $u \neq \widetilde{u}$ in $\mathcal{U}$*

$$\Delta_y \Delta_x \ln f_{Y|X,U}(u) \neq \Delta_y \Delta_x \ln f_{Y|X,U}(\widetilde{u})$$

where $\Delta_y \Delta_x \ln f_{Y|X,U}(u)$ is defined as [3]

$$
\begin{aligned}
\Delta_y \Delta_x \ln f_{Y|X,U}(u) &= \left[\ln f_{Y|X,U}(y|x,u) - \ln f_{Y|X,U}(\overline{y}|x,u)\right] \\
&\quad - \left[\ln f_{Y|X,U}(y|\overline{x},u) - \ln f_{Y|X,U}(\overline{y}|\overline{x},u)\right].
\end{aligned}
$$

The first part of Assumption 3 is directly testable from the data. The second part of Assumption 3 imposes restrictions on the model, which rules out the case where $\ln f_{Y|X,U}$ is additively separable in X and U. Nevertheless, we will show below that Assumption 3(ii) is also testable. That means Assumption 3 is testable from the data given the conditional independence in Assumption 1. Given the matrix invertibility, we may have

$$
\begin{aligned}
\mathbf{A} &\equiv M_{W,y,Z,x} M^{-1}_{W,\overline{y},Z,x} \\
&= M_{W|y,U} D_{y|x,U} D^{-1}_{\overline{y}|x,U} M^{-1}_{W|\overline{y},U}.
\end{aligned}
$$

[3] I use the log function only for the purpose of using the double-difference notation.

Similar matrix manipulations lead to

$$\begin{aligned}
\mathbf{B} &\equiv M_{W,\overline{y},Z,\overline{x}} M_{W,y,Z,\overline{x}}^{-1} \\
&= M_{W|\overline{y},U} D_{\overline{y}|\overline{x},U} D_{y|\overline{x},U}^{-1} M_{W|y,U}^{-1}.
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
\mathbf{AB} &= M_{W|y,U} D_{y|x,U} D_{\overline{y}|x,U}^{-1} D_{\overline{y}|\overline{x},U} D_{y|\overline{x},U}^{-1} M_{W|y,U}^{-1} \\
&\equiv M_{W|y,U} D_{y,\overline{y},x,\overline{x},U} M_{W|y,U}^{-1},
\end{aligned} \tag{7}$$

where

$$D_{y,\overline{y},x,\overline{x},U} = \begin{bmatrix} \exp\left(\Delta_y \Delta_x \ln f_{Y|X,U}(1)\right) & 0 & 0 \\ 0 & ... & 0 \\ 0 & 0 & \exp\left(\Delta_y \Delta_x \ln f_{Y|X,U}(K)\right) \end{bmatrix} \tag{8}$$

Notice that Equation (7) implies an eigenvalue-eigenvector decomposition of a directly estimable matrix **AB**. The second part of Assumption 3 means that all the eigenvalues are distinctive. Because the eigenvalues and eigenvectors are directly estimable from the observed matrices, Assumption 3 is testable from the data. Given that the eigen-decomposition has distinctive eigenvalues, the eigenvectors in $M_{W|y,U}$ are identified up to the permutation of the values of U. To pin down the ordering in one of the decompositions, we impose a normalization assumption as follows:

Assumption 4 *There is a $y_1 \in \mathcal{Y}$ such that i) for any $y \in \mathcal{Y}$, there exists a $(x, \overline{x}, y_1)$ satisfying Assumption 3; and ii) $E[W|Y = y_1, U = u]$ is increasing in u.*

Other normalization assumptions can be found in Hu (2008). In applications where possible values of U doesn't matter, Assumption 4 is not necessary.

Finally, we have identified $f_{W|Y,U}(\cdot|y,\cdot)$ for all y, and further identify distributions $f_{Y|X,U}$ and $f_{Z,X,U}$. We summarize the results as follows:

Theorem 1 *Under assumptions 1, 2, 3, and 4, the joint distribution of four variables $f_{W,Y,Z,X}$ uniquely determines the joint distribution of five variables $f_{W,Y,Z,X,U}$, which satisfies*

$$f_{W,Y,Z,X,U} = f_{W|Y,U} f_{Y|X,U} f_{Z,X,U} \tag{9}$$

Proof: See Appendix.

The constructive proof of Theorem 1 implies that it is a global nonparametric point identification result. We not only identify the complete causal model itself through $f_{Y|X,U}$ but also the joint distribution of the explanatory variables $f(X,U)$. Therefore, it is possible for researchers to integrate out U to estimate ATE and ATET and compare with other approaches.

4 An Alternative Self-Report Procedure

In the case where X is an observed characteristics as in Cases 3 and 4, X will affect all the survey answers and the first part of Assumption 1 may not hold. To be specific, the first part of Assumption 1 contains two restrictions

$$f_{W|Y,X,Z,U} = f_{W|Y,X,U} = f_{W|Y,U}. \tag{10}$$

The first step requires that the self-report procedure doesn't interfere with the causality in the model. The second step requires that the observed explanatory variable X has no impact on the measurement W. The second step may be too strong when X is an individual characteristics. Therefore, we propose another measurement procedure to generate repeated measurements before the outcome is realized. Instead of take a measure of U after the outcome is realized, we may take another measurement Z' of U before the outcome is realized as in Figure 3. The new measurement is supposed to satisfy the assumptions as follows:

Assumption 5 *(Conditional independence) The two measurements, Z and Z', satisfy:*

$$f_{Z|Z',X,U} = f_{Z|X,U}, \tag{11}$$

$$f_{Y|Z,Z',X,U} = f_{Y|X,U}. \tag{12}$$

Again, we assume for simplicity,

Assumption 6 *The two measurements, Z' and Z, and the unobservable U share the same known support $\mathcal{U} = \{1, 2, ..., K\}$.*

This assumption implies that the measurements through surveys will not interfere with the relationship between Y and (X, U), and that the two measurements are independent of each other conditional on (X, U). Given that we may take the surveys at two different times, it is reasonable to assume that this conditional independence. Assumption 2 then implies

$$\begin{aligned} f_{Y,Z',Z,X,U} &= f_{Y|Z',Z,X,U} f_{Z'|Z,X,U} f_{Z,X,U} \\ &= f_{Y|X,U} f_{Z'|X,U} f_{Z,X,U}. \end{aligned} \tag{13}$$

Given $X = x$, three variables Y, Z', and Z are independent conditional on U so that we can use the seminal identification result in Hu (2008) to show that $f_{Y,Z',Z,X,U}$ is identified. We

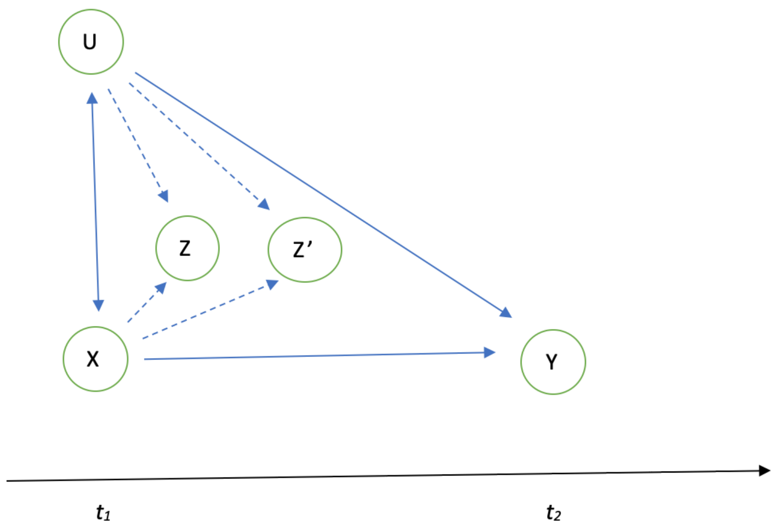

Figure 3: Repeated measurements of an unobservable in a causal model.

define for any fixed (y, x)

$$
\begin{aligned}
M_{y,Z',Z,x} &= [f_{Y,Z',Z,X}(y,i,j,x)]_{i=1,2,...,K;j=1,2,...,K} && (14) \\
M_{Z',Z,x} &= [f_{Z',Z,X}(i,j,x)]_{i=1,2,...,K;j=1,2,...,K} && (15) \\
M_{Z'|,x,U} &= \left[f_{Z'|X,U}(i|x,j)\right]_{i=1,2,...,K;j=1,2,...,K} .
\end{aligned}
$$

Equation (13) is equivalent to

$$
M_{y,Z',Z,x} = M_{Z'|,x,U} D_{y|x,U} (M_{Z,x,U})^T \tag{16}
$$

Similarly,

$$
M_{Z',Z,x} = M_{Z'|,x,U} (M_{Z,x,U})^T \tag{17}
$$

We assume

Assumption 7 *(Invertibility) for any x, $M_{Z',Z,x}$ is invertible.*

To eliminate $M_{Z,x,U}$, we follow Hu (2008) to show

$$
M_{y,Z',Z,x} M_{Z',Z,x}^{-1} = M_{Z'|,x,U} D_{y|x,U} M_{Z'|,x,U}^{-1} \tag{18}
$$

The right hand side is an eigenvalue-eigenvector decomposition of the directly estimable matrix on the left hand side. In order to achieve distinctive eigenvalues, we assume

Assumption 8 *for any $x \in \mathcal{X}$, there exist a $y \in \mathcal{Y}$, such that $f_{Y|X,U}(y|x,u) \neq f_{Y|X,U}(y|x,\overline{u})$ for any $u \neq \overline{u}$ in $\mathcal{U}$.*

Assumption 8 guarantees that all the eigenvectors in columns of $M_{Z'|,x,U}$ are identified. Therefore, $M_{Z'|,x,U}$ is identified up to the permutation of columns. In order to pin down that ordering, we may impose a normalization assumption as follows:

Assumption 9 *For any $x \in \mathcal{X}$, $E[Z'|X = x, U = u]$ is increasing in u.*

We summarize the results as follows:

Theorem 2 *Under Assumptions 5, 6, 7, 8 and 9, the joint distribution of four variables $f_{Y,Z',Z,X}$ uniquely determines the joint distribution of five variables $f_{Y,Z',Z,X,U}$, which satisfies*

$$f_{Y,Z',Z,X,U} = f_{Y|X,U} f_{Z'|X,U} f_{Z,X,U}. \tag{19}$$

Proof: See Appendix.

This identification result is a direct application of Hu (2008). Theorem 2 again implies that it is a global nonparametric point identification of the joint distribution of Y, X, and U. Therefore, the treatment effects can also be identified and estimated accordingly.

5 Summary

This paper presents a novel self-report approach to identify causal models with unobservables. It shows that using a carefully designed self-report procedure researchers are able to identify the complete causal model through the joint distribution of the observables and the unobservables. Given the powerful tools provided in the measurement error literature, it is ready to use this self-report approach to estimate the complete causal model as in physics and chemistry. This paper focuses on a global nonparametric point identification result, but the identification result can be extended in different directions. First, it will be interesting to know how survey can be carefully designed to provide more useful measurements, just as researchers search for better instruments to obtain more accurate measurements in physics and chemistry. Second, one may explore partial identification of the complete model when some of the conditional independence assumptions are relaxed. Third, the estimation of the complete model is straightforward when all the assumptions are satisfied. When the measurements don't contain enough information, for example, the support of measurements is smaller than that of the unobservable, it would be useful to develop partial estimation and inferences in that case.

6 Appendix

Proof of Theorem 1: We start with the joint distribution of four variables $f_{W,Y,Z,X}$. The conditional independence in Assumptions 1 and 2 implies that

$$\begin{aligned} f_{W,Y,Z,X} &= \sum_U f_{W,Y,Z,X,U} \\ &= \sum_U f_{W|Y,Z,X,U} f_{Y|Z,X,U} f_{Z,X,U} \\ &= \sum_U f_{W|Y,U} f_{Y|X,U} f_{Z,X,U}. \end{aligned} \tag{20}$$

For any $(y, x) \in \mathcal{Y}_t \times \mathcal{X}$, we define matrices as follows,

$$\begin{aligned} M_{W,y,Z,x} &= [f_{W,Y,Z,X}(i, y, j, x)]_{i=1,2,\ldots,K;j=1,2,\ldots,K} \\ M_{W|y,U} &= \left[f_{W|Y,U}(i|y, j)\right]_{i=1,2,\ldots,K;j=1,2,\ldots,K}. \\ M_{Z,x,U} &= [f_{Z,X,U}(i, x, j)]_{i=1,2,\ldots,K;j=1,2,\ldots,K}. \\ D_{y|x,U} &= \begin{bmatrix} f_{Y|X,U}(y|x, 1) & 0 & 0 \\ 0 & \ldots & 0 \\ 0 & 0 & f_{Y|X,U}(y|x, K) \end{bmatrix} \end{aligned}$$

Equation (20) is then equivalent to

$$M_{W,y,Z,x} = M_{W|y,U} D_{y|x,U} (M_{Z,x,U})^T \tag{21}$$

A useful observation is that y and x only appear together in the diagonal matrix $D_{y|x,U}$. Therefore, we may consider different values of (y, x) as follows: for (y, x), $(\overline{y}, x)$, $(\overline{y}, \overline{x})$ $(y, \overline{x})$,

$$\begin{aligned} M_{W,y,Z,x} &= M_{W|y,U} \quad D_{y|x,U} \quad (M_{Z,x,U})^T \\ M_{W,\overline{y},Z,x} &= M_{W|\overline{y},U} \quad D_{\overline{y}|x,U} \quad (M_{Z,x,U})^T \\ M_{W,\overline{y},Z,\overline{x}} &= M_{W|\overline{y},U} \quad D_{\overline{y}|\overline{x},U} \quad (M_{Z,\overline{x},U})^T \\ M_{W,y,Z,\overline{x}} &= M_{W|y,U} \quad D_{y|\overline{x},U} \quad (M_{Z,\overline{x},U})^T \end{aligned}$$

Assumption 3 guarantees that for any $y \in \mathcal{Y}$, there exist four matrices on the LHS, which are invertible. And there are four common matrices on the RHS. Therefore, we eliminate $M_{Z,x,U}$ by considering

$$\begin{aligned} \mathbf{A} &\equiv M_{W,y,Z,x} M_{W,\overline{y},Z,x}^{-1} \\ &= M_{W|y,U} D_{y|x,U} D_{\overline{y}|x,U}^{-1} M_{W|\overline{y},U}^{-1}. \end{aligned}$$

Similar matrix manipulations eliminates $M_{Z,\overline{x},U}$ as follows:

$$\begin{aligned}
\mathbf{B} &\equiv M_{W,\overline{y},Z,\overline{x}} M^{-1}_{W,y,Z,\overline{x}} \\
&= M_{W|\overline{y},U} D_{\overline{y}|\overline{x},U} D^{-1}_{y|\overline{x},U} M^{-1}_{W|y,U}.
\end{aligned}$$

Finally, we eliminate $M_{W|\overline{y},U}$ to obtain

$$\begin{aligned}
\mathbf{AB} &= M_{W|y,U} D_{y|x,U} D^{-1}_{\overline{y}|x,U} D_{\overline{y}|\overline{x},U} D^{-1}_{y|\overline{x},U} M^{-1}_{W|y,U} \\
&\equiv M_{W|y,U} D_{y,\overline{y},x,\overline{x},U} M^{-1}_{W|y,U}, \qquad (22)
\end{aligned}$$

where

$$D_{y,\overline{y},x,\overline{x},U} = \begin{bmatrix} \exp\left(\Delta_y \Delta_x \ln f_{Y|X,U}(1)\right) & 0 & 0 \\ 0 & ... & 0 \\ 0 & 0 & \exp\left(\Delta_y \Delta_x \ln f_{Y|X,U}(K)\right) \end{bmatrix}$$

Notice that Equation (22) implies an eigenvalue-eigenvector decomposition of matrix $\mathbf{AB}$, which only contains the joint distribution $f(W,Y,Z,X)$. The eigenvalues are diagonal entries in matrix $D_{y,\overline{y},x,\overline{x},U}$. The eigenvectors are columns in matrix $M_{W|y,U}$, which is a conditional distribution of W given y and a possible value of U. Therefore, the eigenvectors are automatically normalized because all the entries are nonnegative and sum up to 1. The second part of Assumption 3 guarantees that all the eigenvalues are distinctive, which implies that all the corresponding eigenvectors are uniquely determined. Given that the decomposition in Equation (22) has distinctive eigenvalues, the eigenvectors in $M_{W|y,U}$ are identified up to the permutation of the possible values of U. For any $Y = y$ and $X = x$, the unknown distribution $f_{Y|X,U} f_{Z,X,U}$ can be identified from

$$D_{y|x,U} (M_{Z,x,U})^T = M^{-1}_{W|y,U} M_{W,y,Z,x}. \qquad (23)$$

Because Assumption 3 holds for any $y \in \mathcal{Y}$, we have identified distribution $f_{W,Y,Z,X,U}$ satisfying

$$f_{W,Y,Z,X,U} = f_{W|Y,U} f_{Y|X,U} f_{Z,X,U} \qquad (24)$$

up to the permutation of the possible values of U.

If we need to pin down the values of U in $f_{W,Y,Z,X,U}$, we can use Assumption 4. It guarantees that there is a common y_1 such that for any $y \in \mathcal{Y}$ the decomposition above holds with $\overline{y} = y_1$. Therefore, we have

$$\begin{aligned}
\mathbf{A} &\equiv M_{W,y,Z,x} M^{-1}_{W,y_1,Z,x} \\
&= M_{W|y,U} D_{y|x,U} D^{-1}_{y_1|x,U} M^{-1}_{W|y_1,U}.
\end{aligned}$$

and therefore,

$$M_{W|y_1,U} \quad = \quad \mathbf{A}^{-1} M_{W|y,U} D_{y|x,U} D^{-1}_{y_1|x,U}.$$

Each column in $M_{W|y_1,U}$ corresponds to a conditional distribution $f_{W|Y=y_1,U=u}$ for some $u \in \mathcal{U}$. Assumption 4 implies that we can sort the columns in $M_{W|y_1,U}$ by its corresponding conditional mean such that $E[W|Y = y_1, U = u]$ is increasing in u. Therefore, the ordering the columns in $M_{W|y,U}$ is also determined. Therefore, the joint distribution of four variables $f_{W,Y,Z,X}$ uniquely determines the joint distribution of five variables $f_{W,Y,Z,X,U}$. *Q.E.D.*

Proof of Theorem 2: We start with Equation (13)

$$\begin{aligned} f_{Y,Z',Z,X,U} &= f_{Y|Z',Z,X,U} f_{Z'|Z,X,U} f_{Z,X,U} \\ &= f_{Y|X,U} f_{Z'|X,U} f_{Z,X,U}. \end{aligned} \tag{25}$$

We define for any fixed (y, x)

$$M_{y,Z',Z,x} \quad = \quad [f_{Y,Z',Z,X}(y,i,j,x)]_{i=1,2,\dots,K;j=1,2,\dots,K} \tag{26}$$

$$M_{Z',Z,x} \quad = \quad [f_{Z',Z,X}(i,j,x)]_{i=1,2,\dots,K;j=1,2,\dots,K} \tag{27}$$

$$M_{Z'|,x,U} \quad = \quad \left[f_{Z'|X,U}(i|x,j)\right]_{i=1,2,\dots,K;j=1,2,\dots,K}.$$

Equation (13) is equivalent to

$$M_{y,Z',Z,x} = M_{Z'|,x,U} D_{y|x,U} (M_{Z,x,U})^T \tag{28}$$

Similarly, we can show

$$M_{Z',Z,x} = M_{Z'|,x,U} (M_{Z,x,U})^T \tag{29}$$

Assumption 7 implies that

$$M_{y,Z',Z,x} M^{-1}_{Z',Z,x} = M_{Z'|,x,U} D_{y|x,U} M^{-1}_{Z'|,x,U} \tag{30}$$

The left hand side is composed of observed matrices. The right hand side forms an eigenvalue-eigenvector decomposition, where each diagonal element in $D_{y|x,U}$ is an eigenvalue of the matrix on the left hand side and each corresponding column in $M_{Z'|,x,U}$ is an eigenvector. Assumption 8 guarantees that all the eigenvalues are distinctive so that each corresponding eigenvector in columns of $M_{Z'|,x,U}$ are uniquely identified. Therefore, $M_{Z'|,x,U}$ is identified up to the permutation of columns. Each eigenvector is a conditional distribution, and therefore, contains non-negative elements, which add up to one. Assumption 9 pins down the permutation by ordering the conditional expectations corresponding to the eigenvectors. That identifies $M_{Z'|,x,U}$ and $D_{y|x,U}$ for all (x, y). $M_{Z,x,U}$ may then be identified from $M_{Z',Z,x}$. That means all the distributions in Equation (13) are identified. *Q.E.D.*

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